

ON A PENROSE-LIKE INEQUALITY IN DIMENSIONS LESS THAN EIGHT

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ABSTRACT. On an asymptotically flat manifold M^n with nonnegative scalar curvature, with outer minimizing boundary Σ , we prove a Penrose-like inequality in dimensions $n < 8$, under suitable assumptions on the mean curvature and the scalar curvature of Σ .

1. INTRODUCTION AND STATEMENT OF RESULTS

The Riemannian Penrose inequality is a fundamental inequality in mathematical general relativity. It gives a lower bound for the total mass of an asymptotically flat manifold M with nonnegative scalar curvature, in terms of the area of the boundary ∂M , provided ∂M is a minimal hypersurface that is outer minimizing in M . In this paper, we prove a “Penrose-like” inequality in the case where ∂M is not a minimal surface. To state precisely, both the Riemannian Penrose inequality and our main theorems, we first review some definitions.

Definition 1.1. *Let $n \geq 3$. A Riemannian manifold M^n is called asymptotically flat (with one end) if there exists a compact set K such that $M \setminus K$ is diffeomorphic to \mathbb{R}^n minus a ball such that, in the coordinate chart coming from the standard coordinates on \mathbb{R}^n , the metric h on M^n satisfies*

$$(1.1) \quad h_{ij} = \delta_{ij} + O(|x|^{-p}), \quad \partial h_{ij} = O(|x|^{-p-1}), \quad \partial \partial h_{ij} = O(|x|^{-p-2})$$

for some $p > \frac{n-2}{2}$ and the scalar curvature R_h of h satisfies $R_h = O(|x|^q)$ for some $q > n$. Here ∂ denotes the standard partial differentiation on \mathbb{R}^n .

On an asymptotically flat manifold M^n , the limit

$$\mathfrak{m} = \lim_{r \rightarrow \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} (h_{ij,i} - h_{ii,j}) \nu^j d\sigma$$

exists and is known as the ADM mass ([1]) of M . Here ω_{n-1} is the area of the standard unit $(n-1)$ -sphere in \mathbb{R}^n , $S_r = \{x \mid |x| = r\}$, ν is the Euclidean outward unit normal to S_r , $d\sigma$ is the Euclidean area element on S_r , and summation is implied over repeated indices. Under suitable conditions, it was proved by Bartnik [2] and

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Chruściel [5] independently that \mathfrak{m} is a geometric invariant of M ; in particular, the expression for \mathfrak{m} above is independent of coordinates satisfying (1.1).

Definition 1.2. *Given an asymptotically flat manifold M with boundary Σ , one says that Σ is outer minimizing if it minimizes area among all hypersurfaces in M that enclose Σ .*

Theorem 1.1 (Riemannian Penrose inequality in dimensions less than 8). *Let M^n be an asymptotically flat manifold with nonnegative scalar curvature, with boundary Σ , where $3 \leq n \leq 7$. Suppose Σ is a minimal hypersurface that is outer minimizing in M , then*

$$(1.2) \quad \mathfrak{m} \geq \frac{1}{2} \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}},$$

where \mathfrak{m} is the ADM mass of M and $|\Sigma|$ is the area of Σ . Moreover, equality holds if and only if M is isometric to a spatial Schwarzschild manifold outside its horizon.

When $n = 3$, Huisken and Ilmanen [7, 8] first proved Theorem 1.1 for the case that Σ is connected, using inverse mean curvature flow method, and later Bray [3] proved Theorem 1.1 for the general case in which Σ can have multiple components, using a conformal flow of metrics and the Riemannian positive mass theorem [14, 16]. For higher dimensions $n < 8$, Bray and Lee [4] proved Theorem 1.1 using Bray's conformal flow method from [3].

In this paper, we apply Theorem 1.1 to prove a Penrose-like inequality for manifolds whose boundary is not a minimal hypersurface.

Theorem 1.2. *Let M^n be an asymptotically flat manifold of dimension $3 \leq n \leq 7$ with nonnegative scalar curvature, with connected, outer minimizing boundary Σ . Let g be the induced metric on Σ and H be the mean curvature of Σ . Suppose g and H satisfy*

$$(1.3) \quad \min_{\Sigma} R_g > \frac{n-2}{n-1} \max_{\Sigma} H^2,$$

where R_g is the scalar curvature of (Σ, g) . Then

$$(1.4) \quad \mathfrak{m} \geq \frac{1}{2} \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} \left(1 - \frac{n-2}{n-1} \frac{\max_{\Sigma} H^2}{\min_{\Sigma} R_g} \right),$$

where \mathfrak{m} is the ADM mass of M and $|\Sigma|$ is the area of Σ .

Remark 1.1. If M^n is a spatial Schwarzschild manifold outside a rotationally symmetric sphere of positive mean curvature, then equality in (1.4) holds on M^n .

Remark 1.2. The assumption that Σ is outer minimizing implies $H \geq 0$ necessarily.

If the mean curvature H of $\Sigma = \partial M$ is strictly positive, we have the following related but different result.

Theorem 1.3. *Let M^n be an asymptotically flat manifold of dimension $3 \leq n \leq 7$ with nonnegative scalar curvature, with connected, outer minimizing boundary Σ . Let g be the induced metric on Σ and H be the mean curvature of Σ . Suppose $H > 0$ and*

$$(1.5) \quad R_g - 2H\Delta H^{-1} - \frac{n-2}{n-1}H^2 > 0,$$

where R_g is the scalar curvature of (Σ, g) and Δ is the Laplace-Beltrami operator on (Σ, g) . Then

$$(1.6) \quad \mathfrak{m} \geq \frac{1}{2} \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} (1 - \theta),$$

where $\theta \in (0, 1)$ is the constant given by

$$\theta = \frac{n-2}{n-1} \max_{\Sigma} \frac{H^2}{R_g - 2H\Delta H^{-1}}.$$

Here \mathfrak{m} is the ADM mass of M and $|\Sigma|$ is the area of Σ .

Remark 1.3. Both Theorem 1.2 and Theorem 1.3 are applicable to manifolds whose boundary is a constant mean curvature (CMC) hypersurface satisfying

$$R_g - \frac{n-2}{n-1}H_o^2 > 0,$$

where $H = H_o > 0$ is a constant. In this case, both theorems coincide, and (1.4) and (1.6) become

$$\mathfrak{m} \geq \frac{1}{2} \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} \left(1 - \frac{n-2}{n-1} \frac{H_o^2}{\min_{\Sigma} R_g} \right).$$

Remark 1.4. If $n = 3$, the right side of (1.6) can be compared to the Hawking mass of the 2-surface Σ as follows. Let $d\sigma$ be the induced area element on Σ . By (1.5),

$$\begin{aligned} \frac{1}{16\pi} \int_{\Sigma} H^2 d\sigma &= \frac{1}{16\pi} \int_{\Sigma} \frac{H^2}{R_g - 2H\Delta H^{-1}} (R_g - 2H\Delta H^{-1}) d\sigma \\ &\leq \frac{1}{16\pi} \max_{\Sigma} \frac{H^2}{R_g - 2H\Delta H^{-1}} \int_{\Sigma} \left(R_g - \frac{2}{H^2} |\nabla H|^2 \right) d\sigma \\ &\leq \frac{1}{2} \max_{\Sigma} \frac{H^2}{(R - 2H\Delta H^{-1})} = \theta, \end{aligned}$$

where the last inequality follows by the Gauss-Bonnet Theorem. Thus, when Σ is a 2-surface,

$$\frac{1}{2} \left(\frac{|\Sigma|}{\omega_2} \right)^{\frac{1}{2}} (1 - \theta) = \sqrt{\frac{|\Sigma|}{16\pi}} (1 - \theta) \leq \mathfrak{m}_H(\Sigma),$$

where

$$\mathfrak{m}_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\Sigma} H^2 d\sigma \right)$$

is the Hawking quasi-local mass [6] of Σ in M^3 . In this case, a conclusion $\mathbf{m} \geq \mathbf{m}_H(\Sigma)$, which is stronger than (1.6), would follow from the weak inverse mean curvature flow argument in [8] provided M^3 satisfies certain topological assumptions that guarantee the solution to the weak inverse mean curvature flow starting from Σ remains connected. A similar remark also applies to Theorem 1.2 in dimension $n = 3$.

Remark 1.5. The connectedness assumption of $\Sigma = \partial M^n$ in both Theorems 1.2 and 1.3 is not essential. It is assumed here only for the simpleness of the statement of results. See Theorems 3.1 and 3.2 for the case in which Σ has multiple components.

We now explain the idea of the proof of Theorems 1.2 and 1.3, which is largely inspired by the method of Mantoulidis and Schoen in [10]. Given a suitable metric g on the 2-sphere S^2 , Mantoulidis and Schoen constructed a collar extension of (S^2, g) (cf. [10, Lemma 1.3]), which is a metric γ on the cylinder $T = [0, 1] \times S^2$ satisfying the conditions: γ has nonnegative scalar curvature, the induced metric on $\Sigma_0 = \{0\} \times S^2$ agrees with g , Σ_0 is a minimal surface, the induced metric on $\Sigma_t = \{t\} \times S^2$ gets transformed into a round metric as t increases while the area of Σ_t expands and the mean curvature of Σ_t becomes positive in a controlled fashion. For this reason, we would like to call such an extension (T, γ) an “outer collar extension” of (S^2, g) . Given such an outer collar extension (T, γ) , Mantoulidis and Schoen smoothly attach a suitable spatial Schwarzschild manifold to (T, γ) at Σ_1 to obtain an asymptotically flat manifold which has desired geometry near infinity while having an outer most horizon boundary that is isometric to (S^2, g) . Under an assumption that g has positive Gauss curvature, a similar outer collar extension of (S^2, g) , but with the minimal surface condition replaced by a CMC condition, is given in [13].

In contrast to the use of an outer collar extension as in the above work, we make use of an “inner collar extension” in the present paper. More precisely, given the asymptotically flat manifold (M, g) in Theorems 1.2 and 1.3, we construct a metric γ on a cylinder $T = [0, b] \times \Sigma$ for some $b > 0$ satisfying the conditions: γ has nonnegative scalar curvature, the induced metric on $\Sigma_0 = \{0\} \times \Sigma$ agrees with g , the mean curvature of Σ_0 in (T, γ) with respect to the outward normal agrees with (or is greater than) H , and the area of $\Sigma_s = \{s\} \times \Sigma$ decreases as s increases such that the other end Σ_b becomes a minimal hypersurface with controlled area. We then attach this (T, γ) to the given manifold M along the boundary component $\Sigma_0 = \Sigma$ to obtain an asymptotically flat manifold \hat{M} with an outer minimizing minimal hypersurface boundary Σ_b . The metric on \hat{M} may not be smooth across Σ , but the mean curvature conditions on the two sides of Σ in \hat{M} guarantee that the Riemannian Penrose inequality can still be applied to \hat{M} (cf. [12]), which gives the proof of Theorems 1.2 and Theorems 1.3. In particular, the quantities on the right-hand side of (1.4) and (1.6) are simply determined by the area of Σ_b in (T, γ) .

We note that this idea of constructing an inner collar extension, with one end being a minimal hypersurface, to extend the non-minimal boundary of an asymptotically flat manifold is in spirit similar to the idea behind Bray’s inner mass definition [3].

This paper is organized as follows. In Section 2, we construct a suitable inner collar extension of the boundary data described by a triple (Σ, g, H) . In Section 3, we prove Theorems 1.2 and Theorems 1.3 by attaching the inner collar to the given manifold M and applying the Riemannian Penrose inequality.

2. AN INNER COLLAR EXTENSION

In this section, we use a triple (Σ^{n-1}, g, H) to denote a connected, closed manifold Σ of dimension $n - 1$, a Riemannian metric g on Σ , and a positive function H on Σ . We also let r_o be the area radius of (Σ, g) , defined by

$$r_o = \left(\frac{|\Sigma|_g}{\omega_{n-1}} \right)^{\frac{1}{n-1}}.$$

Similar to the outer collar extension constructed in [13], we construct an inner collar extension for (Σ^{n-1}, g, H) as follows. Given any $m \in (0, \frac{1}{2}r_o^{n-2})$, consider the n -dimensional, spatial Schwarzschild manifold

$$(M_m^S, \gamma_m) = \left((r_m, \infty) \times S^{n-1}, \frac{1}{1 - \frac{2m}{r^{n-2}}} dr^2 + r^2 g_* \right),$$

where $r_m = (2m)^{1/(n-2)}$ and g_* denotes the standard metric on S^{n-1} with volume ω_{n-1} . Making a change of variable $s = \int_{r_m}^r \left(1 - \frac{2m}{r^{n-2}}\right)^{-\frac{1}{2}} dr$, we can write the Schwarzschild manifold as

$$(2.1) \quad (M_m^S, \gamma_m) = ([0, \infty) \times S^{n-1}, ds^2 + u_m^2(s) g_*),$$

where $u_m(s)$ satisfies $u_m(0) = r_m$,

$$(2.2) \quad u'_m(s) = \sqrt{1 - \frac{2m}{u_m(s)^{n-2}}} \quad \text{and} \quad u''_m(s) = (n-2) \frac{m}{u_m(s)^{n-1}}.$$

Since $r_m < r_o$, there exists $s_o > 0$ such that $u_m(s_o) = r_o$. In what follows, we will use the finite region in (M_m^S, γ_m) that is bounded by the horizon $\{s = 0\}$ and the sphere $\{s = s_o\}$, as a model to construct an inner collar extension of (Σ^{n-1}, g, H) .

For $s \in [0, s_o]$, let

$$(2.3) \quad v_m(s) = u_m(s_o - s).$$

Given any smooth positive function $A(x)$ on Σ^{n-1} , define the metric

$$(2.4) \quad \gamma_A = A(x)^2 ds^2 + r_o^{-2} v_m(s)^2 g$$

on the product $\Sigma^{n-1} \times [0, s_o]$. For each $s \in [0, s_o]$, the mean curvature $H(s)$ of $\Sigma_s = \{s\} \times \Sigma$ with respect to $\nu = -\partial_s$ is given by

$$(2.5) \quad H_s(x) = \frac{n-1}{A(x)v_m(s)} \sqrt{1 - \frac{2m}{v_m(s)^{n-2}}}$$

by (2.2) and (2.3). In particular, at $s = s_o$,

$$(2.6) \quad \begin{aligned} H_{s_o}(x) &= \frac{(n-1)}{A(x)r_m} \sqrt{1 - \frac{2m}{r_m^{n-2}}} \\ &= 0. \end{aligned}$$

Now, at $s = 0$, if we want to impose $H_0(x) = H(x)$, we must choose

$$(2.7) \quad A(x) = \frac{n-1}{H(x)r_o} \sqrt{1 - \frac{2m}{r_o^{n-2}}}.$$

With such a choice of $A(x)$, using (2.2) and (2.7), one checks (cf. [13]) that the scalar curvature of γ_A is given by

$$(2.8) \quad \begin{aligned} R_{\gamma_A} &= r_o^2 v_m^{-2} (R_g - (n-1)(n-2)r_o^{-2}A^{-2} - 2A^{-1}\Delta A) \\ &= r_o^2 v_m^{-2} \left(R_g - \frac{n-2}{n-1} H^2 \left(1 - \frac{2m}{r_o^{n-2}} \right)^{-1} - 2H\Delta H^{-1} \right), \end{aligned}$$

where R_g is the scalar curvature of g .

This leads us to the following proposition.

Proposition 2.1. *Given a triple (Σ^{n-1}, g, H) , suppose g and H satisfy*

$$(2.9) \quad R_g - \frac{n-2}{n-1} H^2 - 2H\Delta H^{-1} > 0.$$

Let $\theta \in (0, 1)$ be the constant given by

$$(2.10) \quad \theta = \frac{n-2}{n-1} \max_{\Sigma} \frac{H^2}{R_g - 2H\Delta H^{-1}},$$

and define a constant m and a function $A(x)$ by

$$(2.11) \quad m = \frac{r_o^{n-2}}{2} (1 - \theta) \quad \text{and} \quad A(x) = \frac{n-1}{H(x)r_o} \sqrt{1 - \frac{2m}{r_o^{n-2}}},$$

respectively. Then the metric

$$(2.12) \quad \gamma = A(x)^2 ds^2 + r_o^{-2} v_m(s)^2 g$$

defined on $T = [0, s_o] \times \Sigma$ satisfies

- (i) $R_\gamma \geq 0$, i.e. γ has nonnegative scalar curvature;
- (ii) the induced metric on $\Sigma_0 = \{0\} \times \Sigma$ by γ is g ;
- (iii) the mean curvature of Σ_0 with respect to $-\partial_s$ equals H ;
- (iv) $\Sigma_s = \{s\} \times \Sigma$ has positive mean curvature with respect to $-\partial_s$ for each $s \in [0, s_o]$;
- and
- (v) the other boundary component Σ_{s_o} is a minimal surface whose area satisfies

$$(2.13) \quad \frac{1}{2} \left(\frac{|\Sigma_{s_o}|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} = \frac{1}{2} \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} (1 - \theta).$$

Proof. By (2.8), (2.10) and (2.11), it is clear that we have $R_\gamma \geq 0$, which proves (i); (ii) is evident from the definition of γ and the fact $v_m(0) = r_o$; By (2.7), $H_0 = H$ at Σ_0 which proves (iii); (iv) follows directly from (2.5); The fact Σ_{s_o} is a minimal surface follows from (2.6). Clearly,

$$|\Sigma_{s_o}| = \omega_{n-1} v_m(s_o)^{n-1} = \omega_{n-1} (2m)^{(n-1)/(n-2)}$$

which implies (v) by (2.11). \square

Remark 2.1. When $n = 3$, i.e., Σ is a 2-surface, the inner collar extension constructed above provides a valid *fill-in* of the triple (Σ, g, H) , as considered by Jauregui [9, Definition 3].

When the mean curvature function $H(x)$ is a constant, the following is a direct corollary of Proposition 2.1.

Corollary 2.1. *Given a triple (Σ^{n-1}, g, H_o) where H_o is a positive constant, suppose g and H_o satisfy*

$$(2.14) \quad \min_{\Sigma} R_g > \frac{n-2}{n-1} H_o^2.$$

Let $\theta \in (0, 1)$ be the constant given by

$$(2.15) \quad \theta = \frac{n-2}{n-1} \frac{H_o^2}{\min_{\Sigma} R_g},$$

and define two constants m and A_o by

$$(2.16) \quad m = \frac{r_o^{n-2}}{2} (1 - \theta) \quad \text{and} \quad A_o = \frac{n-1}{H_o r_o} \sqrt{1 - \frac{2m}{r_o^{n-2}}},$$

respectively. Then the metric

$$(2.17) \quad \gamma = A_o^2 ds^2 + r_o^{-2} v_m(s)^2 g$$

defined on $T = [0, s_o] \times \Sigma$ satisfies

- (i) $R_\gamma \geq 0$;
- (ii) the induced metric on $\Sigma_0 = \{0\} \times \Sigma$ by γ is g ;
- (iii) the mean curvature of Σ_0 with respect to $-\partial_s$ equals H_o ;
- (iv) $\Sigma_s = \{s\} \times \Sigma$ has positive constant mean curvature with respect to $-\partial_s$ for each $s \in [0, s_o]$; and
- (v) the other boundary component Σ_{s_o} is a minimal surface whose area satisfies

$$(2.18) \quad \frac{1}{2} \left(\frac{|\Sigma_{s_o}|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} = \frac{1}{2} \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} (1 - \theta).$$

3. PROOF OF THE THEOREMS

We are now in a position to prove Theorems 1.2 and 1.3. The idea behind both proofs is the following: we attach an inner collar extension, as constructed in Section 2, to the given asymptotically flat manifold at its boundary, then apply the Riemannian Penrose inequality (1.2) to the resulting manifold. While such a manifold in general is not smooth where the boundaries are joined, provided that the boundary mean curvature from the inner side dominates that from the outer side (cf. [11, 15]), it is known that the Riemannian Penrose inequality still applies. This is proven in [12] in the case $n = 3$ and the same proof applies in dimensions $3 < n \leq 7$.

Proposition 3.1 (Riemannian Penrose inequality on manifolds with corner along a hypersurface). *Let \hat{M}^n denote a noncompact differentiable manifold of dimension $3 \leq n \leq 7$, with compact boundary Σ_H . Let Σ be an embedded hypersurface in the interior of \hat{M} such that Σ and Σ_H bounds a bounded domain Ω . Suppose \hat{h} is a C^0 metric on \hat{M} satisfying:*

- \hat{h} is smooth on both $\hat{M} \setminus \Omega$ and $\bar{\Omega} = \Omega \cup \Sigma \cup \Sigma_H$;
- $(\hat{M} \setminus \Omega, g)$ is asymptotically flat;
- \hat{h} has nonnegative scalar curvature away from Σ ;
- $H_- \geq H_+$, where H_- and H_+ denote the mean curvature of Σ in $(\bar{\Omega}, \hat{h})$ and $(\hat{M} \setminus \Omega, \hat{h})$, respectively, with respect to the infinity-pointing normal; and
- Σ_H is a minimal hypersurface in $(\bar{\Omega}, \hat{h})$ and Σ_H is outer minimizing in (\hat{M}, \hat{h}) .

Then the Riemannian Penrose inequality holds on (\hat{M}, \hat{h}) , i.e.

$$(3.1) \quad \mathfrak{m} \geq \frac{1}{2} \left(\frac{|\Sigma_H|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}},$$

where \mathfrak{m} is the ADM mass of $(\hat{M} \setminus \Omega, \hat{h})$.

We defer the proof of Proposition 3.1 to Appendix A as it is a repetition of the argument from [12], and now turn to proving the main theorems. We begin with Theorem 1.3.

Proof of Theorem 1.3. Given an asymptotically flat manifold M with boundary Σ satisfying the hypotheses of Theorem 1.3, Proposition 2.1 yields a compact manifold (T, γ) with two boundary components, Σ_0 and Σ_{s_o} . Since the induced metric from γ on Σ_0 , which is $\{0\} \times \Sigma$, equals the metric g on Σ , we can attach (T, γ) to M by matching the Gaussian neighborhood of Σ_0 in (T, γ) to that of Σ in M along $\Sigma_0 = \Sigma$. Denote the resulting manifold by \hat{M} and its metric by \hat{h} . By construction, \hat{h} is Lipschitz across Σ and smooth everywhere else on \hat{M} ; it has nonnegative scalar curvature away from Σ ; and the mean curvature of Σ from both sides in \hat{M} agree. Moreover, $\partial\hat{M} = \Sigma_{s_o}$ is a minimal hypersurface that is outer minimizing in \hat{M} . The outer minimizing property is guaranteed by the fact that Σ is outer minimizing in M and the fact that (T, γ) is foliated by $\Sigma_s = \{s\} \times \Sigma$, $s \in [0, s_o]$, which have positive

mean curvature (with respect to $-\partial_s$, pointing towards infinity). By Proposition 3.1, the Riemannian Penrose inequality (Theorem 1.1) applies to such an \hat{M} to give

$$(3.2) \quad \mathfrak{m} \geq \frac{1}{2} \left(\frac{|\Sigma_{s_o}|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}.$$

Theorem 1.3 now follows from (3.2) and (2.13). \square

Theorem 1.2 follows by an almost identical argument:

Proof of Theorem 1.2. If $H \equiv 0$, (1.4) is the Riemannian Penrose inequality (1.2). So it suffices to assume $\max_{\Sigma} H > 0$. In this case, let (T, γ) be the compact manifold given in Corollary 2.1 with the choice of $H_o = \max_{\Sigma} H$. We attach (T, γ) to M and repeat the previous proof. The only difference now is that the mean curvature of Σ in \hat{M} from the side of (T, γ) is H_o while the mean curvature from the side of M is H . By definition, $H_o \geq H$ everywhere on Σ . By Proposition 3.1, Theorem 1.1 still holds on such an \hat{M} , we therefore have

$$(3.3) \quad \mathfrak{m} \geq \frac{1}{2} \left(\frac{|\Sigma_{s_o}|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}.$$

Theorem 1.2 now follows from (3.3) and (2.18). \square

In Theorems 1.3 and 1.2, the connectedness of Σ is assumed only for the simplicity of the statement of the results. It is clear from the above proof that both theorems have analogues that allow the boundary ∂M to have multiple components. For instance, we have

Theorem 3.1. *Let M^n be an asymptotically flat manifold of dimension $3 \leq n \leq 7$ with nonnegative scalar curvature, with outer minimizing boundary ∂M which has connected components $\Sigma_1, \dots, \Sigma_k$. Let g_i be the induced metric on Σ_i and H_i be the mean curvature of Σ_i , $1 \leq i \leq k$. For each i , suppose g_i and H_i satisfy*

$$(3.4) \quad \frac{n-2}{n-1} \max_{\Sigma} H_i^2 < \min_{\Sigma} R_{g_i},$$

where R_{g_i} is the scalar curvature of (Σ_i, g_i) . Then

$$(3.5) \quad \mathfrak{m} \geq \frac{1}{2} \omega^{-\frac{n-2}{n-1}} \left(\sum_{i=1}^k |\Sigma_i| (1 - \theta_i)^{-\frac{n-2}{n-1}} \right)^{\frac{n-2}{n-1}}.$$

Here \mathfrak{m} is the ADM mass of M , $|\Sigma_i|$ is the area of Σ_i and $\theta_i = \frac{n-2}{n-1} \frac{\max_{\Sigma} H_i^2}{\min_{\Sigma} R_{g_i}}$.

Proof. Let $H_{o,i} = \max_{\Sigma_i} H_i$. If $H_{o,i} = 0$ for all i , then (3.5) follows from (1.2). Without losing generality, we may assume $H_{o,j} > 0$, $1 \leq j \leq l$, for some $1 < l \leq k$. For each such j , let (T_j, γ_j) be the compact manifold given in Corollary 2.1 with the choice of $(\Sigma, g, H_o) = (\Sigma_j, g_j, H_{o,j})$. We then attach each (T_j, γ_j) to M at the corresponding Σ_j to obtain a manifold \hat{M} and proceed as in the proof of Theorem

1.2. The boundary $\partial\hat{M}$ in this case consists of $\tilde{\Sigma}_1, \dots, \tilde{\Sigma}_l, \Sigma_{l+1}, \dots, \Sigma_k$, where $\tilde{\Sigma}_j$ is the boundary component of (T_j, γ_j) other than Σ_j . The application of Theorem 1.1 and (2.18) then shows

$$\begin{aligned}
 \mathfrak{m} &\geq \frac{1}{2} \left(\frac{|\partial\hat{M}|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} \\
 (3.6) \quad &= \frac{1}{2} \omega^{-\frac{n-2}{n-1}} \left(\sum_{j=1}^l |\tilde{\Sigma}_j| + \sum_{i=l+1}^k |\Sigma_i| \right)^{\frac{n-2}{n-1}} \\
 &= \frac{1}{2} \omega^{-\frac{n-2}{n-1}} \left(\sum_{i=1}^k |\Sigma_i| (1 - \theta_i)^{-\frac{n-2}{n-1}} \right)^{\frac{n-2}{n-1}},
 \end{aligned}$$

which proves (3.5). \square

In the above proof, replacing the use of Corollary 2.1 by Proposition 2.1, we obtain the following analogue of Theorem 1.3 allowing disconnected boundary.

Theorem 3.2. *Let M^n be an asymptotically flat manifold of dimension $3 \leq n \leq 7$ with nonnegative scalar curvature, with outer minimizing boundary ∂M which has connected components $\Sigma_1, \dots, \Sigma_k$. Let g_i be the induced metric on Σ_i and H_i be the mean curvature of Σ_i , $1 \leq i \leq k$. For each i , suppose g_i and H_i satisfy*

$$(3.7) \quad R_{g_i} - 2H_i \Delta H_i^{-1} - \frac{n-2}{n-1} H_i^2 > 0,$$

where R_{g_i} is the scalar curvature of (Σ_i, g_i) and Δ denotes the Laplace-Beltrami operator on (Σ_i, g_i) . Then

$$(3.8) \quad \mathfrak{m} \geq \frac{1}{2} \omega^{-\frac{n-2}{n-1}} \left(\sum_{i=1}^k |\Sigma_i| (1 - \theta_i)^{-\frac{n-2}{n-1}} \right)^{\frac{n-2}{n-1}}.$$

Here \mathfrak{m} is the ADM mass of M , $|\Sigma_i|$ is the area of Σ_i and $\theta = \frac{n-2}{n-1} \max_{\Sigma_i} \frac{H_i^2}{R_{g_i} - 2H_i \Delta H_i^{-1}}$.

To end this paper, we comment on the equality case in Theorems 1.2 and 1.3. By the equality case in the Riemannian Penrose inequality (Theorem 1.1), one expects that equality in Theorems 1.2 and 1.3 hold only if (M^n, g) is isometric to part of a spatial Schwarzschild manifold that lies outside a compact hypersurface homologous to the horizon. In the context of the proof of Theorems 1.2 and 1.3, this corresponds to showing that equality in (3.1) would imply the manifold (\hat{M}, \hat{h}) in Proposition 3.1 is isometric to a spatial Schwarzschild manifold that lies outside its horizon. Since our current proof of Proposition 3.1 (see Appendix A below) relies on approximating (\hat{M}, \hat{h}) by a sequence of smooth manifolds, and equality on (\hat{M}, \hat{h}) does not necessarily translate into equality for elements in the approximating sequence, we do not have a rigidity statement in Proposition 3.1.

APPENDIX A

In this appendix we give a proof of Proposition 3.1, which was essentially proved in [12] when $n = 3$. The following proof follows closely arguments on pages 279 - 280 in [12], starting from the proof of Lemma 4 therein and ending at its equation (47).

Proof of Proposition 3.1. We will apply the approximation scheme in [11] to the doubling of (\hat{M}, \hat{h}) across its boundary Σ_H . To be precise, let (\hat{M}_c, \hat{h}_c) be a copy of (\hat{M}, \hat{h}) . We glue (\hat{M}, \hat{h}) and (\hat{M}_c, \hat{h}_c) along their common boundary Σ_H to form a Riemannian manifold $(\widetilde{M}, \widetilde{h})$ that has two asymptotically flat ends. Let Σ_c denote the copy of Σ in \hat{M}_c . The metric \widetilde{h} on \widetilde{M} is smooth away from $S = \Sigma \cup \Sigma_H \cup \Sigma_c$. Because of the condition $H_- \geq H_+$ at Σ and because of the fact that Σ_H is minimal in (\hat{M}, \hat{h}) , we can apply Proposition 3.1 in [11] to $(\widetilde{M}, \widetilde{h})$ at S , followed by the conformal deformation specified in [11, Section 4.1], to get a sequence of smooth asymptotically flat metrics $\{\tilde{h}_k\}$ on \widetilde{M} such that

- \tilde{h}_k has nonnegative scalar curvature;
- $\{\tilde{h}_k\}$ converges uniformly to \tilde{h} in the C^0 -topology; and
- the mass \mathbf{m}_k of \tilde{h}_k converges to the mass \mathbf{m} of \tilde{h} on each end of \widetilde{M} .

Furthermore, as $(\widetilde{M}, \widetilde{h})$ has a reflection isometry (which maps a point $x \in \hat{M}$ to its copy in \hat{M}_c), a careful inspection of the analysis in [11, Section 3] reveals that the metrics $\{\tilde{h}_k\}$ can be chosen so as to preserve this reflection symmetry. (Specifically, this is ensured by choosing the mollifier $\phi(t)$ and cut-off function $\sigma(t)$, in (8) and (9) of [11] respectively, to be even functions.) As a result, Σ_H is a totally geodesic, hence minimal, hypersurface in $(\widetilde{M}, \tilde{h}_k)$ for each k .

At this point we turn our attention to the restriction of \tilde{h}_k to \hat{M} , which we denote by h_k . Given any compact hypersurface $\Sigma' \subset \hat{M}$, let $|\Sigma'|_k$ and $|\Sigma'|$ denote the area of Σ' computed in (\hat{M}, h_k) and (\hat{M}, \hat{h}) , respectively. Let \mathcal{S} be the set of all closed hypersurfaces Σ' in \hat{M} which enclose Σ_H . Since Σ_H is minimal in (\hat{M}, h_k) and the dimension n satisfies $3 \leq n \leq 7$, there exists an element $\Sigma_k \in \mathcal{S}$ such that

$$|\Sigma_k|_k = \inf_{\Sigma' \in \mathcal{S}} |\Sigma'|_k,$$

and hence Σ_k is minimal in (\hat{M}, h_k) . Moreover, as $|\Sigma_k|_k \leq |\Sigma_H|_k$, we have

$$(A.1) \quad \limsup_{k \rightarrow \infty} |\Sigma_k|_k \leq \lim_{k \rightarrow \infty} |\Sigma_H|_k = |\Sigma_H|.$$

In particular, $|\Sigma_k|_k \leq C$ for some $C > 0$ independent on k . This together with the fact that $\{h_k\}$ converges uniformly to \hat{h} in C^0 -topology then implies

$$\lim_{k \rightarrow \infty} (|\Sigma_k|_k - |\Sigma_k|) = 0.$$

Consequently,

$$(A.2) \quad \liminf_{k \rightarrow \infty} |\Sigma_k|_k = \liminf_{k \rightarrow \infty} |\Sigma_k| \geq |\Sigma_H|,$$

where we have used the fact Σ_H is outer minimizing in (\hat{M}, \hat{h}) to obtain the inequality. Thus, it follows from (A.1) and (A.2) that

$$(A.3) \quad \lim_{k \rightarrow \infty} |\Sigma_k|_k = |\Sigma_H|.$$

To finish the proof, let $\hat{M}_k \subset \hat{M}$ denote the region that lies outside Σ_k . Theorem 1.1 applies to (\hat{M}_k, h_k) to give

$$(A.4) \quad \mathfrak{m}_k \geq \frac{1}{2} \left(\frac{|\Sigma_k|_k}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}.$$

Passing to the limit, (3.1) follows from (A.3), (A.4) and the fact $\lim_{k \rightarrow \infty} \mathfrak{m}_k = \mathfrak{m}$. \square

REFERENCES

- [1] Arnowitt, R.; Deser, S., and Misner, C. W., *Coordinate invariance and energy expressions in general relativity*, Phys. Rev., **122** (1961), no. 3, 997–1006.
- [2] Bartnik, R., *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math. **39** (1986), no. 5, 661–693.
- [3] Bray, H. L., *Proof of the Riemannian Penrose inequality using the positive mass theorem*, J. Differential Geom., **59** (2001), no. 2, 177–267.
- [4] Bray, H. L.; Lee, D. A., *On the Riemannian Penrose inequality in dimensions less than eight*, Duke Math. J., **148** (2009), no. 1, 81–106.
- [5] Chruściel, P., *Boundary conditions at spatial infinity from a Hamiltonian point of view*, Topological Properties and Global Structure of Space-Time, Plenum Press, New York, (1986), 49–59.
- [6] Hawking, S. W., *Black holes in general relativity*, Comm. Math. Phys., **25** (1972), no. 2, 152–166.
- [7] Huisken, G.; Ilmanen, T., *The Riemannian Penrose inequality*, Int. Math. Res. Not. **20** (1997), 1045–1058.
- [8] Huisken, G.; Ilmanen, T., *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom., **59** (2001), no. 3, 353–437.
- [9] Jauregui, J., *Fill-ins of nonnegative scalar curvature, static metrics, and quasi-local mass*, Pacific J. Math. **261**(2) (2013), 417–444.
- [10] Mantoulidis, C.; Schoen, R., *On the Bartnik mass of apparent horizons*, Class. Quantum Grav., **32** (2015), no. 20, 205002, 16pp.
- [11] Miao, P., *Positive mass theorem on manifolds admitting corners along a hypersurface*, Adv. Theor. Math. Phys., **6** (2002), no. 6, 1163–1182.
- [12] Miao, P., *On a localized Riemannian Penrose inequality*, Commun. Math. Phys., **292** (2009), no. 1, 271–284.
- [13] Miao, P.; Xie, N.-Q., *On compact 3-manifolds with nonnegative scalar curvature with a CMC boundary component*, preprint, arXiv:1610.07513.
- [14] Schoen, R.; Yau, S.-T., *On the proof of the positive mass conjecture in general relativity*, Commun. Math. Phys. **65** (1979), no. 1, 45–76.
- [15] Shi, Y.-G.; Tam, L.-F., *Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature*, J. Differential Geom. **62** (2002), no.1, 79–125.
- [16] Witten, E., *A new proof of the positive energy theorem*, Commun. Math. Phys. **80** (1981), no. 3, 381–402.

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